Exercise 3.1. Let V be an *n*-dimensional vector space and *L* a matrix. *L* has at least one (non-zero) eigenvector. If L is Hermitian then V has an orthonormal basis consisting of eigenvectors of *L*.

Proof. Let be the characteristic polynomial of *L*. is a polynomial in **. By the Fundamental Theorem of Algebra, where *K* is a complex constant, *r* is a positive integer, , and  are roots of , some possibly with multiplicity  greater than 1. In particular, . By footnote (\*) there is a non-zero vector such that . So is an eigenvalue of *L* with eigenvector . This proves that any matrix *L* has at least one eigenvector.



Now suppose that *L* is Hermitian. Then  is real. WLOG we can assume  is a unit vector, . Define the null space . It is easy to see that *N* is a vector subspace of *L*. Since dim  is a 1‑dimensional subspace, the orthogonal subspace *N* has dimension *n* – 1. Claim *LN N*:

Let . We need to show that . Since *L* is Hermitian, . So we need to show that .:

 ✔

Let  restricted to *N*. Repeating our logic above,  has a real root  that is an eigenvalue of  with corresponding unit eigenvector . Since ,  .

Using the (*n* – 2)-dimensional null space of  as above we generate , and since  also.

Continuing this process we eventually obtain the orthonormal basis . ■

(\*) Suppose we have *n* equations in *n* unknowns:



If det *A* ≠ 0, then  exists. Left multiplying by  yields  as the unique solution for the system. That is,  is the unique solution.

If det *A* = 0, then there is not a unique solution. Since  is still a solution, there must be another (non-zero) solution ; that is, .

Returning our attention to *L*, we have  and need to find  such that . Let . Consider the system of *n* equations in *n* unknowns . So   such that . That is,  and .