Exercise 3.1. Let V be an *n*-dimensional complex vector space and *L* a matrix. *L* has at least one eigenvalue, and each distinct eigenvalue has a corresponding (non-zero) eigenvector. In particular, *L* has at least one eigenvector. If *L* is Hermitian then V has an orthonormal basis consisting of eigenvectors of *L*.

Proof. Let  be the characteristic polynomial of *L*.  is a polynomial in **. By the Fundamental Theorem of Algebra,  where *r* is a positive integer, , and  are roots of , some possibly complex and some possibly with multiplicity  greater than 1. So, for all *i*, . By footnote (\*) there is a non-zero vector  such that . So  is an eigenvalue of *L* with eigenvector . This proves that every matrix *L* has at least one eigenvalue and that each distinct eigenvalue has a corresponding (non-zero) eigenvector.

Now suppose that *L* is Hermitian. Then all eigenvalues  are real. WLOG we can assume  is a unit vector, . Define the null space . It is easy to see that *N* is a vector subspace of *L*. Since dim  is a 1‑dimensional subspace, the orthogonal subspace *N* has dimension *n* – 1. Claim *LN N*:

Let . We need to show that . Since *L* is Hermitian, . So we need to show that :

 ✔

Let  restricted to *N*. Repeating our logic above,  has a real root  that is an eigenvalue of  with corresponding unit eigenvector . Since ,  .

Using the (*n* – 2)-dimensional null space of  as above we generate , and since  also.

Continuing this process we eventually obtain the orthonormal basis . ■

Footnote (\*)

Suppose we have *n* equations in *n* unknowns:



If det *A* = 0, then there is an  such that .

Let . So   such that . That is,  and .